Midterm Part I Problem 1

Consider the integral

$$\int_0^3 \sqrt{9+x^3}\mathrm{d}x. \tag{1}$$

(a)(12 points) Calculate Riemann sum approximation as $n = 3$ equal sub-intervals using both left and right endpoints.(You may leave your answers unsimplified.)

Explain how to decide if each is an underestimate or overestimate.

Solution: When $n = 3$, the length of each sub-interval is $\frac{3}{n} = \frac{3}{3} = 1$, and these three intervals are $[0, 1]$, $[1, 2]$ and $[2, 3]$. Hence the left Riemann sum is

$$L_3 = 1 \cdot \sqrt{9+0^3} + 1 \cdot \sqrt{9+1^3} + 1 \cdot \sqrt{9+2^3}, \tag{2}$$

and the right Riemann sum is

$$R_3 = 1 \cdot \sqrt{9+1^3} + 1 \cdot \sqrt{9+2^3} + 1 \cdot \sqrt{9+3^3}. \tag{3}$$

Write $f(x) = \sqrt{9+x^3}$. Since $f'(x) = \frac{3x^2}{2\sqrt{9+x^3}} > 0$ when $x > 0$, the function f is strictly increasing in the interval $(0, 3)$, hence L_3 is always a lower sum and R_3 is an upper sum, therefore L_3 is an underestimate and R_3 is an overestimate.

(Or if you want to explain more explicitly, you can say $\int_0^3 f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^2 f(x)\mathrm{d}x + \int_2^3 f(x)\mathrm{d}x \geq 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = L_3$ and similarly for R_3.)$\diamond$

(6 points) What value of n is sufficient to guarantee that a Riemann sum approximation with n equal sub-intervals is accurate to within 10^{-3}?

Solution: It's something we didn't cover in discussion section and required some understanding of the Riemann sum formula. In general, the largest possible error of the Riemann sum approximation is the difference of the upper sum and the lower sum, and in this case, the upper sum is the right sum R_n and the lower sum is the left sum L_n, so we must have

$$|R_n - L_n| \leq 10^{-3}, \tag{4}$$

since $R_n \geq L_n$, this is equivalent to $R_n - L_n \leq 10^{-3}$, and we can compute this difference out as:

$$R_n - L_n = \frac{3}{n}\sum_{k=1}^{n} f\left(\frac{3(k-1)}{n}\right) - \frac{3}{n}\sum_{k=1}^{n} f\left(\frac{3k}{n}\right) = \frac{3}{n}(f(3) - f(0)) = \frac{9}{n}. \tag{5}$$

Therefore condition (4) is the same as $\frac{9}{n} \leq 10^{-3}$, and hence $n \geq 9 \times 10^3 = 9000$.$\diamond$

(b)(6 points) Explain why the integral equals

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{9 + \left(\frac{3k}{n}\right)^3}\left(\frac{3}{n}\right). \tag{6}$$

Solution: From theorem 4 of the textbook, we know that

$$\int_0^3 \sqrt{9+x^3}\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \tag{7}$$

where $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $x_k = 0 + k\Delta x = \frac{3k}{n}$ and $f(x) = \sqrt{9+x^3}$. Therefore RHS in (7) is equal to

$$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{9+\left(\frac{3k}{n}\right)^3}\left(\frac{3}{n}\right), \tag{8}$$

which is exactly the limit given in the problem.

For this problem, you need at least to relate all the terms in this given limit to the general formula presented in the textbook, but you don't need to mention the number of the theorem.◇